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THE AXISYMMETRIC CONTACT PROBLEM TAKING INTO ACCOUNT TRANSIENT HEAT PRODUCTION DUE TO SLIDING FRICTION†

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The contact problem of the sliding of a solid heat insulator with a plane surface along the boundary of an axisymmetric elastic body is considered, taking into account heat release and the thermal distortion of the boundary of the deformable body due to friction. It is assumed that the shear stresses have no effect on the value of the contact pressures, which enables the problem to be investigated in an axisymmetric formulation. The solution is constructed in two stages: first the form of the thermally distorted surface is determined using known expressions, obtained by Carslaw and Jaeger and also by Barber, and then the contact condition is considered taking into account the elastic displacements and distortion of the form of the surface due to heating, and the integral equation of the problem for determining the unknown contact pressures is derived. The latter equation is solved numerically by approximating the unknown contact pressures by a piecewise-constant function. © 2003 Elsevier Ltd. All rights reserved.

Plane contact problems of thermoelasticity were considered in [1–3], taking into account transient heat production due to friction. The corresponding axisymmetric problem was formulated for the first time by Barber [4]. However, the solution was obtained on the assumption that the distribution of the contact pressure is described by Hertz' formulae throughout the whole interaction process. It is shown below that with this assumption the error when determining the radius of the contact area is 22%, and the problem is solved without this limitation.

1. FORMULATION OF THE PROBLEM

We will consider the contact problem for two semi-infinite bodies, one of which slides along the surface of the other at a constant velocity V and is pressed into it by a force P (Fig. 1). We will assume that the surface of the fixed half-space is slightly curved with a radius of curvature R, and that the sliding is accompanied by the production of heat in the contact area in the form of a heat flux directed into the moving body

$$q(r,t) = fVp(r,t)H(a(t)-r), \quad t>0 \tag{1.1}$$

(p is the contact pressure, a is the radius of the contact circle, f is the coefficient of friction, H is the Heaviside function and r is the radial coordinate on the surface of the half-space $z \geqslant 0$). We will assume that the moving body is a solid heat insulator, the surface of the elastic body outside the contact area is free from external forces and is thermally insulated, and we will neglect the effect of shear forces on the value of the contact pressure.

In this formulation, the contact problem will be axisymmetric. We will represent its solution in the form of the superposition of the solutions of two problems:

(1) the thermoelasticity problem – the determination of the temperature stresses and strains in a heat conducting elastic half-space $z \geqslant 0$ due to its heating by the heat flux (1.1);

(2) the contact problem of thermoelasticity – the determination of the contact stresses when massive bodies are compressed, and the surface of one of them is thermally deformed.

The problem of thermoelasticity. The temperature of the surface of the half-space, due to the fact that it is heated by heat flux (1.1), is equal to [5]

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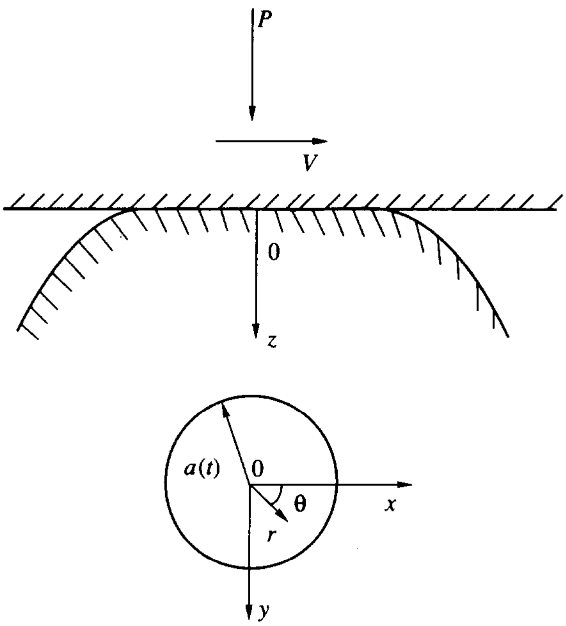

Fig. 1

$$T(r,t) = \frac{1}{4\rho c(\pi k)^{3/2}}\int_0^t\int_0^{a(\tau)}\int_0^{2\pi} q(s,\tau)\exp[-S(r,\theta,s,k(t-\tau))]\frac{s\,d\theta\,ds\,d\tau}{(t-\tau)^{3/2}}, \quad r \geq 0, \quad t > 0$$
$$S(r,\theta,s,k(t-\tau)) = \frac{r^2 - 2rs\cos\theta + s^2}{4k(t-\tau)} \tag{1.2}$$

(k is the thermal diffusivity, ρ is the density and c is the specific heat capacity), and the corresponding thermal normal displacement has the form [6]

$$u_z^t(r,t) = -\frac{\delta}{4\pi}\int_0^t\int_0^{a(\tau)}\int_0^{2\pi} q(s,\tau)\Phi\left(\frac{3}{2};\,2;\,-S(r,\theta,s,k(t-\tau))\right)\frac{s\,d\theta\,ds\,d\tau}{t-\tau}, \quad r \geq 0, \quad t > 0 \tag{1.3}$$

($\delta = \alpha(1+\nu)/\lambda$ is the coefficient of thermal distortion, $\lambda = \rho kc$ is the thermal conductivity, α is the coefficient of linear thermal expansion and Φ is the degenerate hypergeometric function).

The contact problem of thermoelasticity. We will consider the contact problem of thermoelasticity in the quasistatic formulation. In this case the normal displacement u_z^e of the surface of the half-space, due to the action of the contact pressure $p(r,t)H(a(t)-r)$, $t > 0$, is equal to [7]

$$u_z^e(r,t) = \frac{1-\nu}{\mu}\int_0^{a(t)} p(s,t)L(r,s)\,ds, \quad r \geq 0, \quad t > 0 \tag{1.4}$$
$$L(r,s) = \frac{2s}{r+s}\mathbf{K}\left(\frac{4rs}{(r+s)^2}\right)$$

($\mathbf{K}(\cdot)$ is the complete elliptic integral of the first kind, μ is the shear modulus and ν is Poisson's ratio).

The total normal displacement of points of the surface of the elastic half-space will be equal to

$$u_z(r,t) = u_z^e(r,t) + u_z^t(r,t), \quad r \geq 0, \quad t > 0 \tag{1.5}$$

The contact condition of the bodies has the form

$$g(r,t) = u_z(r,t) - \Delta(t) + r^2/(2R) = 0, \quad r \leq a(t), \quad t > 0 \tag{1.6}$$

($\Delta(t)$ is the convergence of the bodies).

We will introduce the following dimensionless variables

$$r^* = \frac{r}{a_{cr}}, \quad s^* = \frac{s}{a_{cr}}, \quad t^* = \frac{tk}{a_{cr}^2}, \quad \tau^* = \frac{\tau k}{a_{cr}^2}, \quad a^* = \frac{a}{a_{cr}}$$
$$p^* = \frac{pa_{cr}^2}{P}, \quad T^* = \frac{T}{T_0}, \quad q^* = \frac{qa_{cr}^2}{fVP}, \quad a_0 = \frac{a(0)}{a_{cr}}, \quad \Delta_0 = \frac{\Delta}{\Delta(0)} \tag{1.7}$$

The values of $a(0)$ and $\Delta(0)$, corresponding to the solution of the isothermal Hertz problem, are as follows [7]:

$$a(0) = \left[\frac{3PR(1-\nu)}{8\mu}\right]^{1/3}, \quad \Delta(0) = \frac{a^2(0)}{R}$$

The quantity

$$a_{cr} = \frac{\pi\lambda(1-\nu)}{1.566\alpha\mu fV(1+\nu)} \tag{1.8}$$

is the limiting (critical) value of the radius of the contact area when the temperature field (1.2) reaches a steady state and there is an unlimited increase ($P \to \infty$) in the indenting force [8]. Note that this limit does not exist in the isothermal case mentioned above.

In the centre of the contact circle the value of the steady temperature is [4]

$$T_0 = \frac{3fVP}{8\lambda a_{cr}} \tag{1.9}$$

Substituting expressions (1.3) and (1.4) into relations (1.5) and (1.6) and changing to the dimensionless variables (1.7) (the asterisk is henceforth omitted), we obtain the integral equation of the problem

$$\frac{1}{\pi}\int_0^{a(t)} q(s,t)L(r,s) -$$
$$-0.16\int_0^t\int_0^{a(\tau)}\int_0^{2\pi} q(s,\tau)\Phi\left[\frac{3}{2};2;-S(r,\theta,s,t-\tau)\right]\frac{sd\theta dsd\tau}{t-\tau} = \frac{3}{8a_0}\left[\Delta_0(t)-\frac{r^2}{2a_0^2}\right] \tag{1.10}$$
$$r \le a(t), \quad t > 0$$

The condition of equilibrium of the body

$$2\pi\int_0^{a(t)} p(s,t)sds = 1 \tag{1.11}$$

must be added to integral equation (1.10) as well as the physical inequalities

$$p(r,t) \ge 0 \ \text{ when } \ r \le a(t); \quad p(r,t) > 0 \ \text{ when } \ r > a(t); \quad t > 0 \tag{1.12}$$

which serve to determine the radius of the contact area $a(t)$.

Moreover, according to notation (1.7), from relation (1.2) we obtain the following expression for the dimensionless temperature in the contact area

$$T(r,t) = \frac{2}{3\pi^{3/2}}\int_0^t\int_0^{a(\tau)}\int_0^{2\pi} q(s,\tau)\exp[-S(r,\theta,s,t-\tau)]\frac{sd\theta dsd\tau}{(t-\tau)^{3/2}}, \quad r \ge 0, \quad t > 0 \tag{1.13}$$

Taking formula (1.1) into account, written in dimensionless form

$$q(r,t) = p(r,t) \tag{1.14}$$

system of equations (1.10), (1.11) enables us to determine the contact pressure $p(r, t)$, the heat flux $q(r,t)$ and the convergence of the bodies $\Delta_0(t)$. Its solution depends on the single dimensionless parameter a_0, characterizing the ratio of the values of the radii of the contact area at the initial instant of time ($t = 0$) and at the final instant of time ($t \to \infty$).

2. NUMERICAL SOLUTION OF THE PROBLEM

We will construct a numerical solution of integral equation (1.10) with conditions (1.11) and (1.12) by the piecewise-constant approximation method [9]. For this purpose, we will divide the time interval $[0, t]$ by points $0 = t_0 < t_1 < \dots < t_l = t$ into l parts of length $\delta t = t/l$. We will partition the contact area by concentric circles with radii $0 = a_0 < a_1 < \dots a_{n-1} < a_n = a(t)$ into n rings of width $\delta a = a/n$. We will assume that the heat flux is constant and equal to q_{ij} in each time-space region $[t_{j-1}, t_j] \times [a_{i-1}, a_i]$. Then, at the instant of time $t = t_l$ we obtain a discrete analogue of Eqs (1.10) and (1.11) in the form

$$\sum_{i=1}^{n} q_{il}\left(\frac{1}{\pi}b_{ik} - 1.28\pi c_{ilkl}\right) = \frac{3}{8a_0}\left[\Delta_0(t_l) - \frac{r_k^2}{2a_0^2}\right] + 1.28\pi\sum_{i=1}^{n}\sum_{j=1}^{l-1} q_{ij}c_{ijkl}, \quad k = 1, 2, \dots, n \tag{2.1}$$

$$2\pi\delta a \sum_{i=1}^{n} q_{il}r_i = 1 \tag{2.2}$$

Here

$$b_{ik} = F_0(r_k, a_i) - F_0(r_k, a_{i-1})$$

$$c_{ijkl} = \begin{cases} \varsigma^{-}_{ijkl1} - \varsigma^{+}_{ijkl1}, & j \neq l \\ -\varsigma^{+}_{ijkl1}, & j = l \end{cases}$$

$$\varsigma^{\pm}_{ijklm} = t^{\pm}_{jl}[(A^{\pm}_{ijl})^2 F_m(R^{\pm}_{kjl}, A^{\pm}_{ijl}) - (A^{\pm}_{i-1,jl})^2 F_m(R^{\pm}_{kjl}, A^{\pm}_{i-1,jl})], \quad m = 1, 2$$

$$R^{\pm}_{kjl} = \frac{r_k}{2(t^{\pm}_{jl})^{1/2}}, \quad A^{\pm}_{ijl} = \frac{a_i}{2(t^{\pm}_{jl})^{1/2}}, \quad t^{\pm}_{jl} = \left(l - j \pm \frac{1}{2}\right)\delta t$$

$$r_k = a_k - \frac{1}{2}\delta a, \quad a_i = i\delta a; \quad k = 1, 2, \dots, n; \quad i = 1, 2, \dots, n; \quad j = 1, 2, \dots, l$$

$$F_0(r, a) = \begin{cases} 2a\mathbf{E}(r/a), & r \leq a \\ 2e[\mathbf{E}(a/r) - (1 - a^2/r^2)\mathbf{K}(a/r)], & r > a \end{cases}$$

$$F_1(R, A) = \begin{cases} \ln(A/2) + (R^2/A^2 + \gamma)/2 + C_1(R, A), & R < A \\ \ln(R/2) + (1 + \gamma)/2 + C_2(R, A), & R \geq A \end{cases}$$

$$G_m(R, A) = \sum_{i=1}^{\infty} \frac{(2i+1)!!}{(2i+2)!!i!i} H_{im}(R, A), \quad m = 1, 2$$

$$H_{i1}(R, A) = (-A^2)^i \sum_{j=0}^{i} (C^i_j)^2 \frac{(R/A)^{2j}}{(i-j+1)}$$

$$H_{i2}(R, A) = (-R^2)^i \sum_{j=0}^{i} (C^i_j)^2 \frac{(A/R)^{2j}}{(j+1)}, \quad C^i_j = \frac{i}{(i-j)!j!}$$

($\gamma = 0.577216$ is Euler's constant and $\mathbf{E}(\cdot)$ is the complete elliptic integral of the second kind). The function $F_0(r, a)$ is written in accordance with the well-known results in [7], and the form of the function $F_1(R, A)$ was obtained previously in [10].

The system of $n + 1$ linear algebraic equations (2.1), (2.2) serves to determine the required quantities q_{il} $(i = 1, 2, \dots, n)$ and $\Delta_0(t_l)$. After solving it we obtain the contact pressure from formula (1.14).

The radius of the contact area $a(t_l)$ is found using inequalities (1.12). If the radius is determined inaccurately, there will be negative contact pressure at points of the contact area, or inter-penetration of the materials of the bodies will occur. We eliminate contact points of the first form and introduce contact points of the second form and repeat the procedure. Calculations showed that five iterations are sufficient to obtain a relative error of less than 1% when calculating $a(t_l)$.

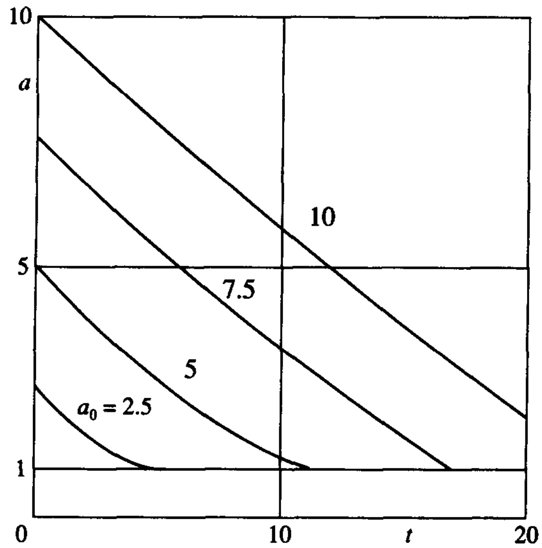

Fig. 2

We obtain the temperature at the contact from relation (1.13), written in discrete form [10]

$$T(r_k, t_l) = 2\sum_{i=1}^{n}\sum_{j=1}^{l} q_{ij}d_{ijkl}$$

$$d_{ijkl} = \begin{cases} \varsigma^{+}_{ijkl2} - \varsigma^{-}_{ijkl2}, & j \neq l \\ \varsigma^{+}_{ijkl1}, & j = l \end{cases}$$

$$F_2(R, A) = \begin{cases} 2(\pi A)^{-1}\mathbf{E}(R/A) - D_1(R, A), & R < A \\ 2R(\pi A^2)^{-1}[\mathbf{E}(A/R) - (1 - A^2/R^2)\mathbf{K}(A/R)] - D_2(R, A), & R \geq A \end{cases}$$

$$D_m(R, A) = \pi^{1/2}\sum_{i=0}^{\infty}\frac{1}{i!(2i+1)}H_{im}(R, A), \quad m = 1, 2$$

The functions $H_{im}(R, A)$ have the form (2.3)

3. RESULTS OF CALCULATIONS

The variation of the dimensionless radius of the contact area $a(t)$ for different values of the dimensionless parameter a_0 is shown in Fig. 2. It was established that the radius of the contact area throughout almost the whole of the transient interaction process decreases linearly, in agreement with the equation $a(t) = a_0 - 0.425t$, obtained previously in [11] by asymptotic analysis. When the system reaches a steady state the radius of the contact area reaches a critical value. Note that the approximate method of solving contact problems taking transient heat production due to friction into account, based on the assumption that the contact pressure distribution has an elliptic form, has not enabled a stationary solution to be found. It was shown in [4, 11] that $a \to 0.783$ as $t \to \infty$. Hence, this approximate method of solving the class of problems considered can be used for small values of the interaction time of the bodies. The value of this time interval depends on the input parameter a_0, representing the ratio of the radii of the contact area at the beginning of the interaction ($t = 0$) and at the end ($t \to \infty$). As a_0 increases the time interval in which the approximate solutions are applicable increases.

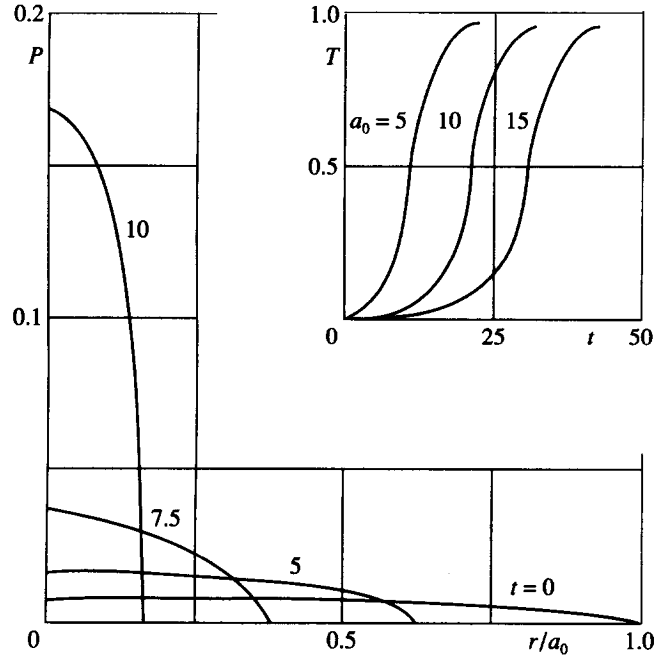

Fig. 3

Diagrams of the dimensionless contact pressure p when $a_0 = 5$ at different instants of time are shown in Fig. 3. Due to the reduction in the size of the contact area, localization of the pressure is observed. In this case the form of its distribution is close to that of the Hertz pressure.

The variation with time of the dimensionless temperature T at the centre of the area of heating ($r = 0$) is shown in the upper right-hand part of Fig. 3 for different values of a_0. It can be seen that the temperature at the contact is fairly low during the whole warmup process and only close to the steady state does a sharp increase in this temperature occur, i.e. in agreement with the notation (1.7), and the true temperature tends to the value T_0 of the form (1.9).

The results of this research show that if the heat production as a result of friction forces at a frictional sliding contact is taken into account there is a considerable redistribution of the contact stresses (compared with the isothermal case). As a result of warmup, thermal distortion of the surface of the elastic body occurs, which leads to a reduction in the contact area. As a consequence of this, contact areas with a high pressure and a high temperature are formed, and on these areas the temperature may considerably exceed the limits permissible for this friction pair, which, in turn, may lead to local adhesion, splitting or other structural changes.

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